Generalization of inequality involving sum with abs of differences.
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Let $a_1, a_2, \ldots, a_m$ be real numbers such that $\sum_{i=1}^{m} a_i^2 = 1$ and $m \geq 3$. Then

$$\sum_{1 \leq i < j \leq m} |a_i - a_j| \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor \sqrt{2}\,.$$

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Solution.

Due symmetry of constraint and $\sum_{1 \leq i < j \leq m} |a_i - a_j|$ we may assume that $a_1 \geq a_2 \geq \ldots \geq a_m$.

Then $\sum_{1 \leq i < j \leq m} |a_i - a_j| = \sum_{1 \leq i < j \leq m} (a_i - a_j) = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} a_i - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} a_j = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} a_i - \sum_{j=2}^{m} \sum_{i=1}^{j-1} a_j =$
$\sum_{i=1}^{m-1} a_i(m-i) - \sum_{j=2}^{m} a_j(j-1) = \sum_{i=1}^{m-1} a_i(m-i) - \sum_{i=2}^{m} a_i(i-1) = (m-1)(a_1 - a_m) +$
$\sum_{i=2}^{m-1} a_i(m-i) - \sum_{i=2}^{m-1} a_i(i-1) = (m-1)(a_1 - a_m) + \sum_{i=2}^{m-1} a_i(m-2i+1)$.

Consider 2 cases:

1. If $m = 2n+1$ then $\sum_{i=2}^{m-1} a_i(m-2i+1) = 2\sum_{i=2}^{2n} a_i(n+1-i) = 2\sum_{i=2}^{n} a_i(n+1-i) +$
$2\sum_{i=n+1}^{2n} a_i(n+1-i) = 2\sum_{i=2}^{n} a_i(n+1-i) - 2\sum_{i=n+1}^{2n} a_i(n+1-i) =$
$2\sum_{i=2}^{n} a_i(n+1-i) - 2\sum_{i=n+2}^{2n} a_i(n+1-i) = 2\sum_{i=2}^{n} a_i(n+1-i) - 2\sum_{i=2}^{n} a_{2n-i+2}(n+1-i) =$
$2\sum_{i=2}^{n}(n+1-i)(a_i - a_{2n-i+2})$ (new index of summation $k := 2n+2-i$ in $\sum_{i=n+2}^{2n} a_i(n+1-i)$).

Thus, if $m = 2n+1$ then $\sum_{1 \leq i < j \leq 2n+1} |a_i - a_j| = 2n(a_1 - a_{2n+1}) + 2\sum_{i=2}^{n}(n+1-i)(a_i - a_{2n-i+2}) \leq$
$2n(a_1 - a_{2n+1}) + 2\sum_{i=2}^{n}(n+1-i)(a_1 - a_{2n+1}) = 2(a_1 - a_{2n+1})\sum_{i=1}^{n}(n+1-i) =$
$n(n+1)(a_1 - a_{2n+1}) = n(n+1)|a_1 - a_{2n+1}| \leq n(n+1)\sqrt{2(a_1^2 + a_{2n+1}^2)} \leq n(n+1)\sqrt{2}$
because $a_1^2 + a_{2n+1}^2 \leq 1$. Equality occurs if $a_1 = -a_{2n+1} = \dfrac{1}{\sqrt{2}}$ and
$a_2 = a_3 = \ldots = a_{2n} = 0$.

2. If $m = 2n$ then $\sum_{i=2}^{m-1} a_i(m-2i+1) = \sum_{i=2}^{2n-1} a_i(2(n-i)+1) =$
$\sum_{i=2}^{n} a_i(2(n-i)+1) + \sum_{i=n+1}^{2n-1} a_i(2(n-i)+1) = \sum_{i=2}^{n} a_i(2(n-i)+1) - \sum_{i=2}^{n} a_{2n+1-i}(2(n-i)+1) =$
$\sum_{i=2}^{n}(2(n-i)+1)(a_i - a_{2n+1-i})$.

Thus, if $m = 2n$ then $\sum_{1 \leq i < j \leq 2n+1} |a_i - a_j| = (2n-1)(a_1 - a_{2n}) + \sum_{i=2}^{n}(2(n-i)+1)(a_i - a_{2n+1-i}) \leq$

$\sum_{i=1}^{n}(2(n-i)+1)(a_1-a_{2n}) = (a_1-a_{2n})\sum_{i=1}^{n}(2(n-i)+1) = n^2(a_1-a_{2n}) = n^2|a_1-a_{2n}| \le$ $n^2\sqrt{2(a_1^2+a_{2n}^2)} \le n^2\sqrt{2}$ with equality if $a_1 = -a_{2n} = \dfrac{1}{\sqrt{2}}$ and $a_2 = a_3 = \ldots = a_{2n-1} = 0$.

Obtained in both cases results can represented by one inequality:

$$\sum_{1\le i<j\le m}|a_i-a_j| \le \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor \sqrt{2},$$

with equality if $a_1 = -a_m = \dfrac{1}{\sqrt{2}}$ and $a_2 = a_3 = \ldots = a_{m-1} = 0$.